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STABILITY OF NONLINEAR CIRCUITS

by

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Abstract

The stability of the various states of equilibrium of circuits containing a nonlinear element is dependent upon the dynamic current-voltage characteristics of the element and cannot be determined from the static characteristic alone. The speed with which triggering takes place between two states of equilibrium is governed by the manner in which the slope of the dynamic characteristic depends upon rates of change of current and voltage and therefore upon the reactive circuit elements.

More than one state of equilibrium may exist in a circuit containing an element that exhibits a negative resistance over a portion of its ŏperating range. This may be readily shown graphically for the simple series circuit of Fig. 1, in which N represents a nonlinear circuit element. Possible equilibrium values of current and voltage may be determined by a current-voltage diagram similar to that used in the graphical analysis of vacuum tubes. Thus in Fig. 2, curve a represents the static current-voltage characteristic of the nonlinear element and the line MN is the "load line" corresponding to the series resistance R and the applied voltage E_{bb}. Equilibrium values of current and voltage are those corresponding to the intersections of the current-voltage characteristic with the load line. Figure 2 shows clearly that if the current-voltage characteristic of the nonlinear element includes a portion with negative slope greater in magnitude than the slope of the load line, three intersections, and therefore three equilibrium values of current and voltage are possible.

Experimentally it is ordinarily found that the equilibrium corresponding to intersection 2 on the negative-resistance section of the characteristic curve is unstable, whereas those corresponding to intersections 1 and 3 are stable. By means of careless reasoning, it is possible to "prove" with the aid of Fig. 2, that point 2 should be unstable. An argument similar to the following is usually used:[1]

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[1] H. J. Reich, "Theory and Applications of Electron Tubes", page 350, McGraw-Hill Book Co., Inc., New York, 2nd Edition, 1944.

> If the current and voltage have values corresponding to point 2, any small increase of current due to any cause is accompanied by a decrease of voltage across the element. More voltage is thus made available to send current through the resistance, and so the current rises further. The action is cumulative, the current rising until point 1 is reached. Any further increase of current above that corresponding to point 1 would necessarily be accompanied by an increase of voltage across the tube. The voltage across the resistance would therefore have to fall, which could be true only if the current became smaller. Hence the current would return to the value corresponding to point 1. A similar analysis shows that if the current is initially that corresponding to point 2, any small initial decrease of current becomes cumulative, and so the current falls to the value at point 3.

Equally careless reasoning may be employed to show that point 2 seemingly corresponds to stable equilibrium. Thus, if at every value of current the voltage across element N is added to the IR drop across the resistance R, the curve of Fig. 3 is obtained. Since equilibrium values of current are those at which the

sum of the voltage across N and that across R are equal to the supply voltage E_{bb}, possible equilibrium values of current are those at which the curve intersects the abscissa corresponding to E_{bb}. It now becomes apparent that if the regions immediately surrounding the points 1, 2, and 3 are considered, no difference in behavior should be expected in the vicinity of these points. Furthermore, if the current increases from its value at point 2, the voltage required to maintain the flow of current exceeds that available from the supply, whereas if the current decreases, the supply voltage exceeds that required to maintain current flow, and the current might be expected to return to its initial value corresponding to point 2. One might, therefore be tempted to conclude that point 2 corresponds to stable equilibrium, whereas experimental evidence indicates that it usually corresponds to unstable equilibrium.

The solution to this seeming dilemma lies in the fact that the characteristic curve plotted in Fig. 2 is a static curve, whereas a departure from a point of equilibrium necessarily involves changes of current and voltages. Since the rate of change of current and voltage from an equilibrium value may be very great, any predictions as to stability must be based upon dynamic current-voltage characteristics, of which there are an infinite number. The manner in which the current varies with voltage in the vicinity of any point depends upon the location of the point in the current-voltage diagram and upon the time rate of change of voltage.

Although it is neither possible nor profitable to predict the behavior of a particular circuit by graphical means, certain fairly obvious facts may be used to determine the general criterion for instability. In the first place, if the nonlinear element contains no reactive elements or their equivalent (i.e., if the current carriers are assumed to have no mass), the dynamic characteristic must always be identical with the static characteristic, regardless of the rate of change of current or of the voltage across the nonlinear element. Similarly, if R represents a pure resistance, the dynamic load line is identical with the static load line, regardless of the rate of change of current or voltage. Then, since the sum of the voltages in the series circuit must always be equal to the applied voltage, and this requirement cannot be satisfied for values of current differing even infinitesimally from the values at the equilibrium points, the currents cannot depart from these values. All three intersections in the diagrams of Figs. 2 and 3 must therefore be stable under these assumed conditions.

Negative-resistance regions in a current-voltage characteristic are the result of some form of positive feedback (electrical or other) and are present only when the open-circuit loop amplification of the system exceeds unity. When the amplification is produced by vacuum tubes, the circuit always contains reactances, principally capacitive, that cause the amplification to fall off as the frequency, or rate of change of voltage, is increased. The slope of the dynamic characteristic in a negative-resistance region therefore becomes less steep as the rate of change of voltage is increased. Thus, in the vicinity of intersection 2, progressive increase of rate of change of voltage might cause the slope of the dynamic characteristic for increasing or decreasing voltage to change progressively through values indicated by curves a, b, and c and eventually to become positive.[1] (Note that if the dynamic characteristic is obtained by

[1] Although the circuit of Fig. 1 is a series circuit, and the voltage across N might be considered a function of the current and the rate of change of current, N is a voltage-stable device, rather than a current-stable device. Because the characteristic curve has a maximum and a minimum, the static and dynamic characteristics cannot be obtained by varying the current, and dynamic characteristics at constant rate of change of current cannot be determined directly, since the rate of change must be zero at the maximum and minimum values. Rate of change of voltage, on the other hand, may be maintained constant in obtaining dynamic characteristics.

varying the voltage periodically, presence of reactive elements in the amplifier causes the alternating component of current to be out of phase with the alternating component of voltage. The dynamic characteristic will therefore have a hysteresis loop.)

If the voltages and the current initially have the values corresponding to the equilibrium point 2 and the voltage across N starts to increase at such a rate that the resulting dynamic characteristic at point 2 is tangent to the load line, a positive voltage increment across N is accompanied by an equal negative increment of the voltage across R and a current change can occur without violation of the requirement that the sum of the voltages across N and R must be equal to the supply voltage. If, as the voltage across N increases, the rate of change of

current, and hence of voltage across N adjusts itself so that at each new point the corresponding dynamic characteristic is tangent to the load line, the current can decrease along the load line until point 3 is reached. Once started, the decrease continues, since only in that way can Kirchhoff's law be continuously satisfied. If the rate of rise of voltage across N were to fall below the value necessary to make the slope of the dynamic characteristic equal to the slope of the load line, the sum of the voltage drops would immediately exceed the applied voltage, which is impossible. Once the voltage is increased from the value at point 2, therefore, it increases continuously to the value at point 3, the rate of change at each value of current being such as to make the overall dynamic characteristic coincide with the load line. Similarly, any downward perturbation of voltage from the value at point 2 causes the voltage to decrease continuously along the load line until point 1 is reached.

The general way in which the path of operation is related to dynamic characteristics is shown by Fig. 5. Such a diagram must be carefully interpreted, however, because each dynamic characteristic involved in the transfer from point 2 to point 1 is determined not only by the rate of change of voltage, but also, because of charges stored on capacitors and currents flowing through inductors, upon currents and voltages assumed at previous instants. The characteristic corresponding to a certain rate of increase of voltage is not necessarily the same as that for the same rate of decrease at the same point in the diagram. Actually, since the only requirement that a particular dynamic characteristic need satisfy is that it be tangent to the load line at the point on the path at which the time rate of change of voltage is that corresponding to the characteristic, the exact shape of any dynamic characteristic is of no consequence.

If it were possible to devise an element N for which the slope of the dynamic characteristic in the negative-resistance region increases with increasing rate of change of voltage, point 2 would be stable.

In regions in which the static characteristic curve has positive slope, the slope of a dynamic characteristic corresponding to any rate of change of voltage may differ from that of the static characteristic, but cannot become negative

for nonlinear deviced commonly used.[1] The dynamic characteristic that must be

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[1]This statement may seem to contradict the previous argument used in considering a change of current from point 2 to point 1. The difference, in the vicinity of point 2, between the path followed from 2 to 1 and a dynamic characteristic for a change of current <u>from</u> point 1 is accounted for by voltages across inductances and currents flowing into capacitances within the element N. If it were possible to set up instantaneously the necessary currents and voltages at all points within the element N, the current could be made to return from 1 to 2 and thence to 3.

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followed <u>from</u> points 1 and 3 therefore cannot coincide with the load line, and the current cannot change from the values at these points without violating Kirchhoff's law. Consequently these points are stable.

If R is not a pure resistance, but includes appreciable capacitance or inductance, the slope of the load line for changing voltages may be either greater or less than the slope of the static load line, and the voltage change at every instant takes place along a dynamic load line corresponding to the instantaneous rate of change of voltage. The path followed therefore differs from the static load line, and is in general not linear.

The foregoing analysis also explains the manner in which the current changes when the circuit is made to trigger from point 3 to point 1, or vice versa, by shifting the load line or the characteristic curve. If the current has the value corresponding to point 3 in Fig. 6, for instance, and the supply voltage is gradually decreased, the stable point moves down the right-hand branch of the curve until point 3' is reached, at which the load line is tangent to the curve. Any small change of voltage from the value at 3' will then take place along a dynamic characteristic having a slope that is less negative (or more positive) than the slope of the static characteristic at point 3'. Increase of voltage across N would therefore have to take place along a dynamic characteristic the slope of which cannot be equal to that of the load line. Decrease of voltage, on the other hand, will take place along a dynamic characteristic the slope of which approaches that of the load line as the rate of change of voltage is increased. The rate of change of current, and hence of voltage, therefore adjusts itself at every instant so that the resultant dynamic characteristic coincides with the load line, and the current increases along the load line until point 1' is reached.

This analysis indicates that if there are no reactive elements in series with N, speed of triggering should be increased by increase of resistance R. The larger R is, the flatter is the load line and the more the slope of the dynamic load line must depart from that of the static load line in order that the resulting dynamic characteristic can coincide at every point with the load line, and therefore the greater must be the rate of change of current at each point.

If the resistance R is shunted by a small amount of capacitance, on the other hand, the dynamic load line is steeper than the static load line by an amount that increases as the rate of change of voltage is increased. Increase of current then takes place along a path that lies above the static load line, and decrease takes place along a path that lies below the static load line, as illustrated in Fig. 7. Because these paths differ from the static characteristic curve by smaller amounts than does the static load line, the rate of change of voltage at every instant is less than without the capacitance. The shunt capacitance therefore reduces the speed of triggering. At large values of R, the shunt capacitance prevents increase of speed of triggering with increase of R.

Although only voltage-stable nonlinear elements have been considered in the foregoing analysis, similar reasoning may be applied to current-stable nonlinear elements, i.e., those in which the current is multivalued in one or more ranges of voltage.

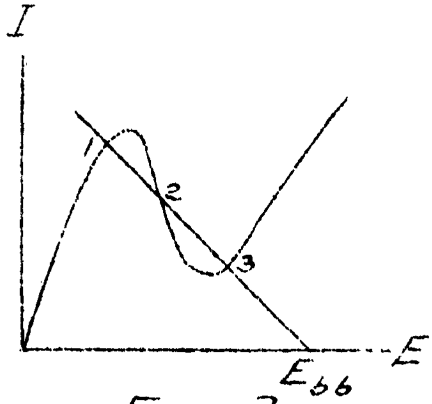

FIG. 2

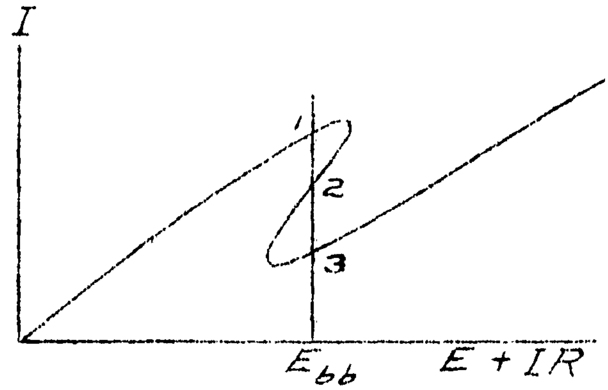

FIG. 3

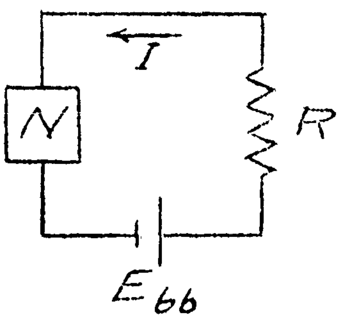

FIG. 1

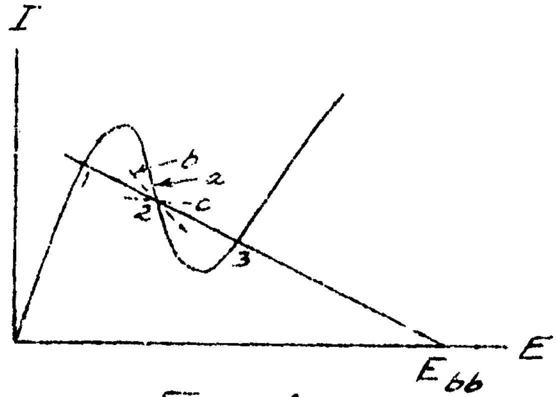

FIG. 4

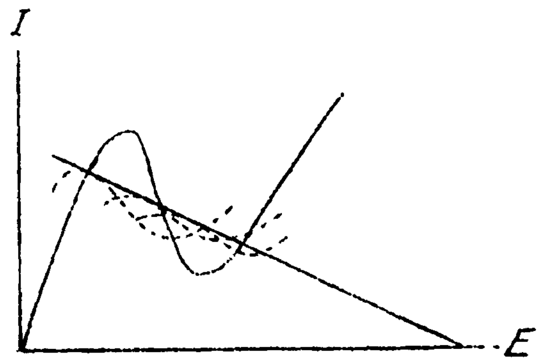

FIG. 5

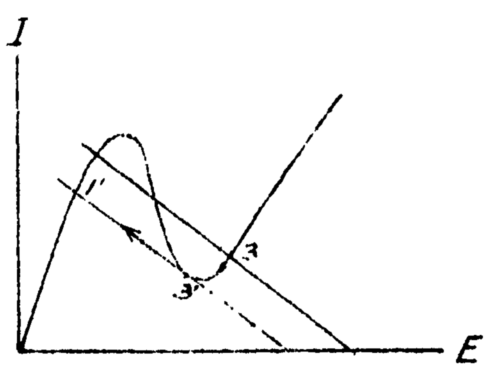

FIG. 6

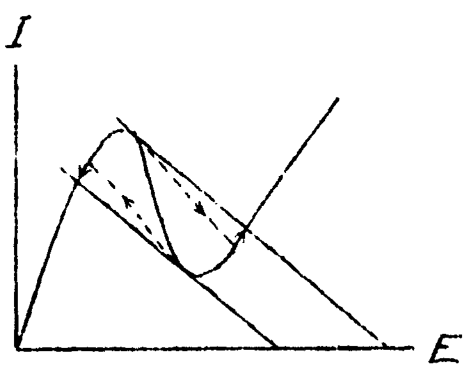

FIG. 7